

# JACOB'S LADDERS, FACTORIZATION AND METAMORPHOSES AS AN APPENDIX TO THE RIEMANN FUNCTIONAL EQUATION FOR $\zeta(s)$ ON THE CRITICAL LINE

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ABSTRACT. In this paper we obtain a new set of metamorphoses of the oscillating Q-system by using the Euler's integral. We split the final state of mentioned metamorphoses into three distinct parts: the signal, the noise and finally appropriate error term. We have also proved that the set of distinct metamorphoses of that class is infinite one.

## 1. INTRODUCTION AND THE FIRST RESULT

1.1. Let us remind the Riemann's functional equation (1859)

$$(1.1) \quad \zeta(1-s) = 2(2\pi)^{-s} \cos \frac{\pi s}{2} \Gamma(s) \zeta(s), \quad s \in \mathbb{C} \setminus \{1\}.$$

*Remark 1.* It is known that L. Euler discovered a formula equivalent to (1.1) for real values of the variable  $s$  in 1749, (comp. [2], pp. 23-26), however the proof in the Euler's work is missing.

Next, if we put

$$(1.2) \quad \chi(s) = \pi^{s-1/2} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} \Rightarrow \chi(s)\chi(1-s) = 1,$$

(comp. [9], pp. 13, 16) then we obtain (see (1.1), (1.2)) that

$$(1.3) \quad \frac{\zeta(1-s)}{\zeta(s)} = \chi(s).$$

*Remark 2.* Now, in connection with (1.3) we use the following (E. Landau, [3], p. 30): the quotient

$$\frac{\zeta(1-s)}{\zeta(s)}$$

is expressed by the *known* function ... (of course, *known* function is such one that is different from  $\zeta(s)$ ).

1.2. However, in the case

$$(1.4) \quad s = \frac{1}{2} + it, \quad 1-s = \frac{1}{2} - it = \left(\frac{1}{2} + it\right)^*$$

we have (comp. (1.2), (1.4)) that

$$\left| \chi\left(\frac{1}{2} + it\right) \right| = 1,$$

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i.e. (see (1.3))

$$(1.5) \quad \left| \frac{\zeta\left(\frac{1}{2} + it\right)}{\zeta\left(\frac{1}{2} + i\gamma\right)} \right| = 1, \quad \forall t \neq \gamma : \zeta\left(\frac{1}{2} + i\gamma\right) = 0,$$

(after continuation for  $t = \gamma$  this is valid for all  $t \in \mathbb{R}$ ).

*Remark 3.* Consequently, on the critical line

$$s = \frac{1}{2} + it$$

the Riemann's functional equation (1.3) gives only the trivial result (1.5) (of course, the main aim of the Riemann's functional equation is the analytic continuation of  $\zeta(s)$  to  $\mathbb{C} \setminus \{1\}$ ).

1.3. It is clear that in this situation our Remark 2 leads us to the following

**Question.** about another method of sampling of the points

$$(1.6) \quad t_2 > t_1 > 0$$

(say) from a corresponding set that the quotient (comp. (1.3), (1.5))

$$(1.7) \quad \left| \frac{\zeta\left(\frac{1}{2} + it_2\right)}{\zeta\left(\frac{1}{2} + it_1\right)} \right|, \quad t_1, t_2 \neq \gamma$$

is expressed by a known function.

There is method giving an answer to the Question – namely one answer from a set of many possibilities – mentioned method is our method of transformation by using the reversely iterated integrals (comp. [7], (4.1) – (4.19)).

1.4. In this paper we obtain, for example, the following

**Formula1.** For every sufficiently big  $L \in \mathbb{N}$  and for every  $U \in (0, \pi)$  there are functions

$$(1.8) \quad \begin{aligned} \alpha_0^4 &= \alpha_0^4(L, U; a, b), \\ \alpha_1^4 &= \alpha_1^4(L, U; a, b), \\ \beta_1^4 &= \beta_1^4(L, U), \\ \alpha_0^4, \alpha_1^4, \beta_1^4 &\neq \gamma : \zeta\left(\frac{1}{2} + i\gamma\right) = 0 \end{aligned}$$

such that

$$(1.9) \quad \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_1^4\right)}{\zeta\left(\frac{1}{2} + i\beta_1^4\right)} \right| \sim \frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U}{2}\right) a + b \cos(\alpha_0^4)}{\sqrt{\frac{a-b}{a+b}} \frac{U}{2}}, \quad L \rightarrow \infty,$$

where

$$(1.10) \quad \begin{aligned} \alpha_0^4 &\in (2\pi L, 2\pi L + U), \\ \alpha_1^4, \beta_1^4 &\in \left(\widehat{2\pi L}, \widehat{2\pi L + U}\right), \\ (2\pi L, 2\pi L + U) &\prec \left(\widehat{2\pi L}, \widehat{2\pi L + U}\right), \end{aligned}$$

and the functions (1.8) have the following property

$$\alpha_1^4 - \alpha_0^4 \sim (1 - c)\pi(2\pi L), \quad L \rightarrow \infty,$$

where  $c$  is the Euler's constant and  $\pi(t)$  is the prime-counting function.

*Remark 4.* The formula (1.9) gives one answer to our Question (comp. (1.7), (1.10)) in the direction outlined in Remark 2.

*Remark 5.* Let us notice explicitly that in the case (1.7), i.e. on the critical line, we may suppose that the *known* function is *every* function, for example

$$f[|\zeta|], \arg \zeta, \dots$$

In this direction, we have obtained the following (see also corresponding formulae in the papers [6] – [8],  $k = 1$ )

**Formula2.**

$$(1.11) \quad \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_1^2\right)}{\zeta\left(\frac{1}{2} + i\beta_1^2\right)} \right| \sim \frac{\sqrt{\zeta(2\sigma)}}{|\zeta[\sigma + i\alpha_0^2(\sigma)]|}, \quad \sigma \geq 1.$$

**Formula3.**

$$(1.12) \quad \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_1^3\right)}{\zeta\left(\frac{1}{2} + i\beta_1^3\right)} \right| \sim \pi^l \sqrt{c_l} \left| \int_0^{\alpha_0^3(T)} \arg \zeta\left(\frac{1}{2} + it\right) dt \right|^{-l}.$$

**Formula4.**

$$(1.13) \quad \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_1^1\right)}{\zeta\left(\frac{1}{2} + i\beta_1^1\right)} \right| \sim \sqrt[4]{\frac{2\pi}{H}} \frac{1}{\sqrt{|\zeta(\frac{1}{2} + i\alpha_0^1)|}}.$$

*Remark 6.* Of course, the results (1.9), (1.11) – (1.13) are based on properties of the Jacob's ladder  $\varphi_1(t)$  as follows:

$$(1.14) \quad \begin{aligned} \alpha_0^4 &= \varphi_1^1(d) = \varphi_1(d) \in (2\pi L, 2\pi L + U), \\ \alpha_1^4 &= \varphi_1^0(d) = d \in (\overbrace{2\pi L}^1, \overbrace{2\pi L + U}^1), \\ \beta_1^4 &= \varphi_1^0(e) = e \in (\overbrace{2\pi L}^1, \overbrace{2\pi L + U}^1), \end{aligned}$$

say (comp. (1.14) and (1.10)).

1.5. Let us remind - for completeness - that Jacob's ladder

$$\varphi_1(t) = \frac{1}{2}\varphi(t)$$

has been introduced in our work [4] (see also [5]), where the function

$$\varphi(t)$$

is arbitrary solution of the non-linear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t) e^{-\frac{2}{x(T)}t} dt = \int_0^T Z^2(t) dt,$$

where each admissible function  $\mu(y)$  generates the solution

$$y = \varphi(T; \mu) = \varphi(T), \quad \mu(y) \geq 7y \ln y.$$

The function  $\varphi_1(t)$  is called the Jacob's ladder according to Jacob's dream in Chumash, Bereishis, 28:12.

*Remark 7.* We have shown (see [4]), by making use of these Jacob's ladders, that the classical Hardy-Littlewood integral (1918)

$$\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt$$

has - in addition to the Hardy-Littlewood expression (and other similar to that one) possessing an unbounded error at  $T \rightarrow \infty$  - the following infinite set of almost exact expressions

$$\begin{aligned} \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt &= \varphi_1(T) \ln \varphi_1(T) + \\ &+ (c - \ln 2\pi) \varphi_1(T) + c_0 + \mathcal{O} \left( \frac{\ln T}{T} \right), \quad T \rightarrow \infty, \end{aligned}$$

where  $c$  is the Euler's constant and  $c_0$  is the constant from the Titchmarsh-Kober-Atkinson formula.

*Remark 8.* The Jacob's ladder  $\varphi_1(t)$  can be interpreted by our formula (see [4])

$$T - \varphi_1(T) \sim (1 - c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T},$$

where  $\pi(T)$  is the prime-counting function, as an asymptotically complementary function to

$$(1 - c)\pi(T)$$

in the following sense

$$\varphi_1(T) + (1 - c)\pi(T) \sim T, \quad T \rightarrow \infty.$$

## 2. FACTORIZATION, OSCILLATING Q-SYSTEM AND ITS METAMORPHOSES AS A GENERIC COMPLEMENT TO THE RIEMANN'S FUNCTIONAL EQUATION ON THE CRITICAL LINE

2.1. The oscillating Q-system was defined in our work [7], (2.1) as follows

$$\begin{aligned} G(x_1, \dots, x_k; y_1, \dots, y_k) &\stackrel{\text{def}}{=} \prod_{r=1}^k \left| \frac{\zeta \left( \frac{1}{2} + ix_r \right)}{\zeta \left( \frac{1}{2} + iy_r \right)} \right| = \\ (2.1) \quad &= \prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(x_r)} \frac{2}{\sqrt{n}} \cos\{\vartheta(x_r) - x_r \ln n\} + R(x_r)}{\sum_{n \leq \tau(y_r)} \frac{2}{\sqrt{n}} \cos\{\vartheta(y_r) - y_r \ln n\} + R(y_r)} \right|, \\ \tau(t) &= \sqrt{\frac{t}{2\pi}}, \quad R(t) = \mathcal{O}(t^{-1/4}), \quad k \leq k_0 \in \mathbb{N}, \end{aligned}$$

for corresponding sets (see [7], (2.2)) of the points

$$(x_1, \dots, x_k), \quad (y_1, \dots, y_k).$$

*Remark 9.* It is clear that the definition relation (2.1) is based on simple generalization

$$\left| \frac{\zeta \left( \frac{1}{2} + ix \right)}{\zeta \left( \frac{1}{2} + iy \right)} \right| \longrightarrow \prod_{r=1}^k \left| \frac{\zeta \left( \frac{1}{2} + ix_r \right)}{\zeta \left( \frac{1}{2} + iy_r \right)} \right|$$

(comp. (1.7), (1.11) – (1.13)).

Let us remind some of the previous results playing the role of the Riemann's functional equation on the critical line.

(A). There are the functions (see [7], (2.5))

$$\begin{aligned}\alpha_r^2 &= \alpha_r^2(\sigma, T, \Theta, k, \epsilon), \quad r = 0, 1, \dots, k, \\ \beta_r^2 &= \beta_r^2(T, \Theta, k), \quad r = 1, \dots, k, \\ \alpha_r^2, \beta_r^2 &\neq \gamma : \quad \zeta\left(\frac{1}{2} + i\gamma\right) = 0,\end{aligned}$$

for admissible

$$\sigma, T, \Theta, k, \epsilon$$

such that the following factorization formula

$$(2.2) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^2\right)}{\zeta\left(\frac{1}{2} + i\beta_r^2\right)} \right| \sim \frac{\sqrt{\zeta(2\sigma)}}{|\zeta[\sigma + i\alpha_0^2(\sigma)]|}, \quad T \rightarrow \infty,$$

(see [7], (2.6)) holds true, i.e. there is following set of metamorphoses of the oscillating Q-system (2.1):

$$(2.3) \quad \begin{aligned} & \prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(\alpha_r^2)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_r^2) - \alpha_r^2 \ln n\} + R(\alpha_r^2)}{\sum_{n \leq \tau(\beta_r^2)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\beta_r^2) - \beta_r^2 \ln n\} + R(\beta_r^2)} \right| \sim \\ & \sim \sqrt{\zeta(2\sigma)} \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{\sigma + i\alpha_0^2(\sigma)} \right|, \quad \sigma > 1 + \epsilon, \quad T \rightarrow \infty \end{aligned}$$

(see [7], (2.6)), where  $\mu(n)$  is the Möbius function.

(B). There are functions

$$\begin{aligned}\alpha_r^3 &= \alpha_r^3(T, l, \epsilon, k), \quad r = 0, 1, \dots, k, \quad l \in \mathbb{N} \\ \beta_r^3 &= \beta_r^3(T, \epsilon, k), \quad r = 1, \dots, k, \\ \alpha_r^3, \beta_r^3 &\neq \gamma : \quad \zeta\left(\frac{1}{2} + i\gamma\right) = 0,\end{aligned}$$

for admissible

$$T, l, \epsilon, k$$

such that the following factorization formula

$$(2.4) \quad \begin{aligned} & \left| \int_0^{\alpha_0^3} \arg \zeta\left(\frac{1}{2} + it\right) dt \right| \sim \\ & \sim \pi c^{\frac{1}{2l}} \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^3\right)}{\zeta\left(\frac{1}{2} + i\beta_r^3\right)} \right|^{-\frac{1}{l}}, \quad T \rightarrow \infty \end{aligned}$$

(see [8], (2.4)) holds true, i.e. there is following set of metamorphoses of the oscillating Q-system (2.1):

$$(2.5) \quad \begin{aligned} & \prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(\alpha_r^3)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_r^3) - \alpha_r^3 \ln n\} + R(\alpha_r^3)}{\sum_{n \leq \tau(\beta_r^3)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\beta_r^3) - \beta_r^3 \ln n\} + R(\beta_r^3)} \right| \sim \\ & \sim \pi^l \sqrt{c_l} \left| \int_0^{\alpha_0^3} \arg \zeta\left(\frac{1}{2} + it\right) dt \right|, \quad T \rightarrow \infty, \end{aligned}$$

(see [8], (4.6)).

(B1). If we rewrite the formula (2.4) as follows

$$\left| \int_{\mu_m}^{\alpha_0^3} \arg \zeta \left( \frac{1}{2} + it \right) dt \right| \sim \pi c_l^{\frac{1}{2l}} \prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(\alpha_r^3)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_r^3) - \alpha_r^3 \ln n\} + R(\alpha_r^3)}{\sum_{n \leq \tau(\beta_r^3)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\beta_r^3) - \beta_r^3 \ln n\} + R(\beta_r^3)} \right|^{-\frac{1}{l}}$$

where

$$m = m(\alpha_0^3), \mu_m < \alpha_0^3 < \mu_{m+1}; S_1(\mu_m) = 0$$

(see [8], (4.11)), then we obtain the set of metamorphoses (2.5) in reverse direction: we begin with

$$\left| \int_0^w \arg \zeta \left( \frac{1}{2} + it \right) dt \right|$$

that is the Aaron staff (say),

$$\longrightarrow \left| \int_{\mu_m}^{\alpha_0^3} \arg \zeta \left( \frac{1}{2} + it \right) dt \right|$$

that is the bud of the Aaron staff (corresponding to  $w = \alpha_0^3$ )

$$\sim \left| \frac{\sum_{n \leq \tau(\alpha_r^3)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_r^3) - \alpha_r^3 \ln n\} + R(\alpha_r^3)}{\sum_{n \leq \tau(\beta_r^3)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\beta_r^3) - \beta_r^3 \ln n\} + R(\beta_r^3)} \right|$$

already metamorphosed into almonds ripened (comp. Chumash, Bamidbar, 17:23).

(C). We have obtained the first set of metamorphoses of the primeval multiform

$$G(x_1, \dots, x_k) = \prod_{r=1}^k \left| \zeta \left( \frac{1}{2} + ix_r \right) \right|$$

in our paper [6]. The corresponding results expressed in terms of oscillating Q-system (2.1) are (see [6], (1.7), (2.5)): the factorization formula

$$(2.6) \quad \prod_{r=1}^k \left| \frac{\zeta \left( \frac{1}{2} + ix \right)}{\zeta \left( \frac{1}{2} + iy \right)} \right| \sim \sqrt[4]{\frac{2\pi}{H}} \frac{1}{|\zeta(\frac{1}{2} + i\alpha_0^1)|}$$

and corresponding set of metamorphoses of the oscillating Q-system

$$(2.7) \quad \prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(\alpha_r^1)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_r^1) - \alpha_r^1 \ln n\} + R(\alpha_r^1)}{\sum_{n \leq \tau(\beta_r^1)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\beta_r^1) - \beta_r^1 \ln n\} + R(\beta_r^1)} \right| \sim \frac{\sqrt[4]{\frac{2\pi}{H}}}{\sqrt{\left| \sum_{n \leq \tau(\alpha_0^1)} \frac{2}{\sqrt{n}} \cos\{\vartheta(\alpha_0^1) - \alpha_0^1 \ln n\} + R(\alpha_0^1) \right|}}, \quad T \rightarrow \infty.$$

(D). Moreover, the sequences

$$\{\alpha_r^n\}_{r=0}^k, \{\beta_r^n\}_{r=1}^k, \quad n = 1, 2, 3$$

have the following universal property:

$$\begin{aligned} \alpha_{r+1}^n - \alpha_r^n &\sim (1-c)\pi(T), \quad r = 0, 1, \dots, k-1, \\ \beta_{r+1}^n - \beta_r^n &\sim (1-c)\pi(T), \quad r = 1, \dots, k-1, \quad k \geq 2, \end{aligned}$$

(comp. (1.6), (1.7), Remark 8 and 9).

### 3. THEOREM; FACTORIZATION AS AN ANALOGUE OF THE RIEMANN'S FUNCTIONAL EQUATION ON THE CRITICAL LINE

3.1. We use the following Euler's integral (see [1], pp. 134, 135)

$$(3.1) \quad \int \frac{d\varphi}{a + b \cos \varphi} = \frac{1}{\sqrt{a^2 - b^2}} \arctan \frac{(a-b) \tan \frac{\varphi}{2}}{\sqrt{a^2 - b^2}}, \quad \varphi \in (0, \pi),$$

$$a + b > 0, a^2 - b^2 > 0 \implies a > |b|.$$

We obtain immediately from (3.1) the following formula

$$(3.2) \quad \begin{aligned} &\int_{2\pi L}^{2\pi L+U} \frac{d\varphi}{a + b \cos \varphi} = \\ &= \frac{1}{a+b} U \frac{\arctan \left( \sqrt{\frac{a-b}{a+b}} \tan \frac{U}{2} \right)}{\sqrt{\frac{a-b}{a+b}} \frac{U}{2}}, \quad L \in \mathbb{Z}, \quad U \in (0, \pi). \end{aligned}$$

3.2. Now, if we use our method of transformation (see [7], (4.1) – (4.19)) in the case of the formula (3.2) then we obtain the following statement.

**Theorem.** Let

$$(3.3) \quad [2\pi L, 2\pi L + U] \longrightarrow [\widehat{\pi L}, \widehat{\pi L + U}], \dots, [\widehat{\pi L}, \widehat{\pi L + U}]$$

where

$$[\widehat{\pi L}, \widehat{\pi L + U}], \quad r = 1, \dots, k, \quad k \leq k_0 \in \mathbb{N}$$

are reversely iterated segments corresponding to the first segment in (3.3) and  $k_0$  be arbitrary and fixed number. Then there is a sufficiently big

$$T_0 = T_0(a, b) > 0$$

such that for every

$$L > \frac{1}{2\pi} T_0$$

and for every admissible  $L, U, k$  there are functions

$$(3.4) \quad \begin{aligned} &\alpha_r^4(L, U, k; a, b), \quad r = 0, 1, \dots, k, \\ &\beta_r^4(L, U, k), \quad r = 1, \dots, k, \\ &\alpha_r^4, \beta_r^4 \neq \gamma: \quad \zeta\left(\frac{1}{2} + i\gamma\right) = 0 \end{aligned}$$

such that

$$(3.5) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^4\right)}{\zeta\left(\frac{1}{2} + i\beta_r^4\right)} \right|^2 \sim \frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U}{2}\right)}{\sqrt{\frac{a-b}{a+b}} \frac{U}{2}} \frac{a + b \cos \alpha_0^4}{a + b}, \quad L \rightarrow \infty,$$

of course

$$G(\alpha^4, \beta^4) \sim \sqrt{(\dots)} \iff \{G(\alpha^4, \beta^4)\}^2 \sim (\dots).$$

Moreover, the sequences

$$\{\alpha_r^4\}_{r=0}^k, \{\beta_r^4\}_{r=1}^k, \quad n = 1, 2, 3$$

have the following properties

$$(3.6) \quad \begin{aligned} 2\pi L &< \alpha_0^4 < \alpha_1^4 < \dots < \alpha_k^4, \\ 2\pi L &< \beta_1^4 < \beta_2^4 < \dots < \beta_k^4, \\ \alpha_0^4 &\in (2\pi L, 2\pi L + U), \\ \alpha_r^4, \beta_r^4 &\in (\overbrace{2\pi L}^r, \overbrace{2\pi L + U}^r), \quad r = 1, 2, \dots, k, \end{aligned}$$

$$(3.7) \quad \begin{aligned} \alpha_{r+1}^4 - \alpha_r^4 &\sim (1 - c)\pi(2\pi L), \quad r = 0, 1, \dots, k - 1, \\ \beta_{r+1}^4 - \beta_r^4 &\sim (1 - c)\pi(2\pi L), \quad r = 1, \dots, k - 1, \quad k \geq 2, \end{aligned}$$

where

$$\pi(T) \sim \frac{T}{\ln T}, \quad T \rightarrow \infty$$

is the prime-counting function and  $c$  is the Euler's constant.

3.3. Now, let us notice the following.

*Remark 10.* The asymptotic behavior of the following sets

$$(3.8) \quad \{\alpha_r^4\}_{r=0}^k$$

is as follows: if  $L \rightarrow \infty$  then the points of every set (3.8) recede unboundedly each from other and all together recede to infinity. Hence, at  $L \rightarrow \infty$  each of the sets (3.8) behaves as one-dimensional Friedmann-Hubble universe.

*Remark 11.* Next, we express the result (3.5) in connection with (1.1), (1.3), Remark 2 and Remark 9 in the form

$$(3.9) \quad \prod_{r=0}^k \left| \zeta\left(\frac{1}{2} + i\alpha_r^4\right) \right| \sim \sqrt{\chi_4(U, \alpha_0^4)} \prod_{r=1}^k \left| \zeta\left(\frac{1}{2} + i\beta_r^4\right) \right|,$$

where  $\chi_4$  stands for the right-hand side of (3.5). It is quite clear that this formula is an analogue of the Riemann's function equation on the critical line.

*Remark 12.* Of course, the first result (1.9) is a particular case of (3.5).



## 4. ON A SET OF METAMORPHOSES THAT CORRESPOND TO THE FORMULA (3.5)

4.1. Let us remind the spectral form of the Riemann-Siegel formula

$$(4.1) \quad \begin{aligned} Z(t) &= \sum_{n \leq \tau(x_r)} \frac{2}{\sqrt{n}} \cos \{t\omega_n(x_r) + \psi(x_r)\} + \mathcal{O}(x_r^{-1/4}), \\ \tau(x_r) &= \sqrt{\frac{x_r}{2\pi}}, \\ t &\in [x_r, x_r + V], \quad V \in (0, x_r^{1/4}), \end{aligned}$$

and similarly for  $x_r \rightarrow y_r$ , where

$$T_0 < 2\pi L < x_r, y_r$$

(see [6], (6.1), comp. [8], (4.4) and Remark 6 *ibid*).

*Remark 13.* We call the expressions

$$(4.2) \quad \frac{2}{\sqrt{n}} \cos \{t\omega_n(x_r) + \psi(x_r)\}, \dots$$

as the Riemann's oscillators with:

(a) the amplitude

$$\frac{2}{\sqrt{n}},$$

(b) the incoherent local phase-constant

$$\psi(x_r) = -\frac{x_r}{2} - \frac{\pi}{8},$$

(c) the non-synchronized local time

$$t = t(x_r) \in [x_r, x_r + V],$$

(d) the local spectrum of the cyclic frequencies

$$\{\omega_n(x_r)\}_{n \leq \tau(x_r)}, \quad \omega_n(x_r) = \ln \frac{\tau(x_r)}{n}, \dots$$

*Remark 14.* Now, we see that the Q-system, where of course

$$(4.3) \quad \left| \zeta \left( \frac{1}{2} + it \right) \right| = |Z(t)|,$$

is really complicated oscillating system (comp. (2.1), (4.1), (4.3)).

4.2. Consequently, we have (see (2.1), (3.5), (4.1), Remark 14 and (4.3)) the following.

**Corollary 1.** The following set of metamorphoses

$$(4.4) \quad \begin{aligned} &\prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(\alpha_r^4)} \frac{2}{\sqrt{n}} \cos \{ \alpha_r^4 \omega_n(\alpha_r^4) + \psi(\alpha_r^4) \} + R(\alpha_r^4)}{\sum_{n \leq \tau(\beta_r^4)} \frac{2}{\sqrt{n}} \cos \{ \beta_r^4 \omega_n(\beta_r^4) + \psi(\beta_r^4) \} + R(\beta_r^4)} \right| \sim \\ &\sim \frac{\arctan \left( \sqrt{\frac{a-b}{a+b}} \tan \frac{U}{2} \right) a + b \cos(\alpha_0^4)}{\sqrt{\frac{a-b}{a+b}} \frac{U}{2} a + b}, \quad L \rightarrow \infty. \end{aligned}$$

corresponds to the factorization formula (3.5).

4.3. Now, let us notice the following.

*Remark 15.* By Theorem, there are control functions (3.4) (Golem's shem) of the set of metamorphoses (4.4) of the oscillating Q-system (2.1), (see also (4.1), (4.3)).

*Remark 16.* The mechanism of metamorphosis is as follows. Let (comp. (3.4) and [7], (2.2))

$$(4.5) \quad \begin{aligned} M_k^3 &= \{\alpha_1^4, \dots, \alpha_k^4\}, \\ M_k^4 &= \{\beta_1^4, \dots, \beta_k^4\}, \end{aligned}$$

where. of course, (comp. [7], (2.12))

$$(4.6) \quad \begin{aligned} M_k^3 &\subset M_k^1 \subset (T_0, +\infty)^k, \\ M_k^4 &\subset M_k^2 \subset (T_0, +\infty)^k. \end{aligned}$$

Now, if we obtain, after random sampling such points (comp. conditions [7], (2.2)) that

$$(4.7) \quad \begin{aligned} (x_1, \dots, x_k) &= (\alpha_1^4, \dots, \alpha_k^4) \subset M_k^3, \\ (y_1, \dots, y_k) &= (\beta_1^4, \dots, \beta_k^4) \subset M_k^4, \end{aligned}$$

(see (4.5), (4.6)), then – at the points (4.7) – the Q-system (2.1) changes its old form (=chrysalis) into its new form (=butterfly) and the last is controlled by the function  $\alpha_0^4$ .

*Remark 17.* Now, it should be clear that the set of metamorphoses of oscillating Q-system also belongs to the family of analogues of the Riemann's functional equation on the critical line.

## 5. ON DECOMPOSITION OF THE RESULT OF METAMORPHOSES (4.4) INTO THREE PARTS: SIGNAL, NOISE AND ERROR TERM

In this section we use the terminology from the theory of signal processing.

5.1. Let us remind (see [7], (4.11)) that

$$\begin{aligned} \tilde{Z}^2(t) &= \frac{|\zeta(\frac{1}{2} + it)|^2}{\omega(t)}, \\ \omega(t) &= \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right) \right\} \ln t. \end{aligned}$$

Since in our case

$$t \longrightarrow 2\pi L,$$

then (comp. [7], (4.11), (4.12))

$$\tilde{Z}^2(\alpha_r^4) = \frac{|\zeta(\frac{1}{2} + i\alpha_r^4)|^2}{\left\{ 1 + \mathcal{O}\left(\frac{\ln \ln L}{\ln L}\right) \right\} \ln t}, \dots$$

*Remark 18.* Consequently, the primary form of the asymptotic formula for metamorphoses (4.4) is as follows

$$(5.1) \quad \prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(\alpha_r^4)} \frac{2}{\sqrt{n}} \cos\{\alpha_r^4 \omega_n(\alpha_r^4) + \psi(\alpha_r^4)\} + R(\alpha_r^4)}{\sum_{n \leq \tau(\beta_r^4)} \frac{2}{\sqrt{n}} \cos\{\beta_r^4 \omega_n(\beta_r^4) + \psi(\beta_r^4)\} + R(\beta_r^4)} \right| \sim$$

$$\sim \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln L}{\ln L}\right) \right\} \frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U}{2}\right)}{\sqrt{\frac{a-b}{a+b}} \frac{U}{2}} \frac{a + b \cos(\alpha_0^4)}{a + b}.$$

Since the last two factors on the right-hand side of (5.1) are bounded functions for all

$$U \in (0, \pi), \quad L > \frac{1}{2\pi} T_0$$

(for all fixed admissible  $k, a, b$ , see (3.1), (3.4)), then we obtain from (5.1) the following.

**Corollary 2.**

$$(5.2) \quad \prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(\alpha_r^4)} \frac{2}{\sqrt{n}} \cos\{\alpha_r^4 \omega_n(\alpha_r^4) + \psi(\alpha_r^4)\} + R(\alpha_r^4)}{\sum_{n \leq \tau(\beta_r^4)} \frac{2}{\sqrt{n}} \cos\{\beta_r^4 \omega_n(\beta_r^4) + \psi(\beta_r^4)\} + R(\beta_r^4)} \right| =$$

$$= \frac{a}{a+b} \frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U}{2}\right)}{\sqrt{\frac{a-b}{a+b}} \frac{U}{2}} +$$

$$+ \frac{b}{a+b} \frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U}{2}\right)}{\sqrt{\frac{a-b}{a+b}} \frac{U}{2}} \cos(\alpha_0^4) +$$

$$+ \mathcal{O}\left(\frac{\ln \ln L}{\ln L}\right), \quad L \rightarrow \infty.$$

5.2. Let us remind (see (3.4)), that

$$(5.3) \quad \alpha_0^4 = \alpha_0^4(L, U, k; a, b) = \alpha_0^4(L, U)$$

for admissible and fixed  $k, a, b$ .

- (a) We see that the first function on the right-hand side of (5.2) is the  $L$ -th member

$$(5.4) \quad f(2\pi L + U) = g_L(U) = \frac{a}{a+b} \frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U}{2}\right)}{\sqrt{\frac{a-b}{a+b}} \frac{U}{2}}, \quad U \in (0, \pi)$$

$$g_L(U) = g_{L'}(U), \quad \forall L, L' > \frac{1}{2\pi} T,$$

$$2\pi L + U \in [2\pi L, 2\pi L + \pi)$$

of the stationary sequence

$$(5.5) \quad \{g_L(U)\}_{L > T_0/2\pi}, \quad U \in (0, \pi).$$

By (5.4), (5.5) the corresponding signal is defined.

Consequently, by the first function on the right-hand side of (5.2) deterministic signal is expressed (see (5.5)).

- (b) The main factor in the second member is the following function

$$\cos(\alpha_0^4), \alpha_0^4 \in \alpha_0^4(L, U),$$

where (comp. (1.14), (5.3))

$$\alpha_0^4 = \varphi_1(d) \in (2\pi L, 2\pi L + U), \quad d = d(L, U),$$

and  $\varphi_1(d)$  is the value of the Jacob's ladder. That is, the distribution of the values

$$\alpha_0^4 \in (2\pi L, 2\pi L + U), \quad U \in (0, \pi)$$

we may suppose as very complicated.

Consequently, the second function we shall characterize as noise – a non-useful part of the signal. The noise may be controlled by variation of the parameter  $b$ ,

$$a > |b|$$

(see (3.1)), i.e. by abatement of  $|b|$ .

- (c) The third function we shall call (fine) error term, since

$$\mathcal{O}\left(\frac{\ln \ln L}{\ln L}\right) \xrightarrow{L \rightarrow \infty} 0.$$

*Remark 19.* Hence, the final state of metamorphoses in (5.2) is split into three parts: signal, noise and (fine) error term.

## 6. THE SET OF DISTINCT METAMORPHOSES IN (4.4)

Of course, there is a point  $U_0 \in (0, \pi)$  such that (see (5.4))

$$g'_L(U)|_{U=U_0} \neq 0, \quad \forall L > \frac{T_0}{2\pi},$$

i.e.

$$(6.1) \quad \begin{aligned} g'_L(U) &\neq 0, \quad U \in O_\delta(U_0) = (U_0 - \delta, U_0 + \delta), \\ \forall U', U'' \in O_\delta(U_0), \quad U' \neq U'' &\Rightarrow g_L(U') \neq g_L(U'') \end{aligned}$$

for suitable  $\delta > 0$ .

Next, we shall suppose that there are such

$$U_1, U_2 \in O_\delta(U_0), \quad U_1 \neq U_2,$$

that

$$(6.2) \quad \begin{aligned} \alpha_r^4(U_1, L) &= \alpha_r^4(U_2, L), \quad r = 0, 1, \dots, k, \\ \beta_r^4(U_1, L) &= \beta_r^4(U_2, L), \quad r = 1, \dots, k \end{aligned}$$

for all

$$L > \tilde{L} > \frac{T_0}{2\pi}.$$

In this case we obtain, by comparison of the formulas (5.1), for  $U_1, U_2$  that

$$\begin{aligned} &\frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U_1}{2}\right)}{\sqrt{\frac{a-b}{a+b}} \frac{U_1}{2}} = \\ &= \left\{1 + \mathcal{O}\left(\frac{\ln \ln L}{\ln L}\right)\right\} \frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U_2}{2}\right)}{\sqrt{\frac{a-b}{a+b}} \frac{U_2}{2}}, \end{aligned}$$

i.e. in the limit case we obtain the equality

$$\frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U_1}{2}\right)}{\sqrt{\frac{a-b}{a+b}} \frac{U_1}{2}} = \frac{\arctan\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{U_2}{2}\right)}{\sqrt{\frac{a-b}{a+b}} \frac{U_2}{2}}$$

that contradicts with (6.1).

Hence, we have that for every

$$U', U'' \in O_\delta(U_0), \quad U' \neq U''$$

there is an infinite subsequence

$$\{\bar{L}\} \subset \{L\}, \quad \bar{L} > \tilde{L}$$

such that (comp. (6.2))

$$(6.3) \quad \begin{aligned} &(\alpha_0^4(U', \bar{L}), \alpha_1^4(U', \bar{L}), \dots, \alpha_k^4(U', \bar{L}), \\ &\beta_1^4(U', \bar{L}), \dots, \beta_k^4(U', \bar{L})) \neq \\ &\neq (\alpha_0^4(U'', \bar{L}), \alpha_1^4(U'', \bar{L}), \dots, \alpha_k^4(U'', \bar{L}), \\ &\beta_1^4(U'', \bar{L}), \dots, \beta_k^4(U'', \bar{L})), \quad \forall \bar{L} \in \{\bar{L}\}. \end{aligned}$$

Consequently, by (6.3) we have the following.

**Corollary 3.** There is an infinite set of distinct metamorphoses (4.4).

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